Lesson 9.1 Series Definitions, Geometric Series, n^{th} Term Test

Factorial Definition: $n! = n(n-1)(n-2)(n-3)\bullet\bullet\bullet 1$

Example 1: Simplify $\dfrac{(n+2)!}{n!} =$

Definition: A **series** is a **sum** of numbers.

An infinite series can be represented as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Example 2: Write out the first five terms of the series $\sum_{n=1}^{\infty} \dfrac{n}{n+1} =$

Examples: Write an expression for the nth term of the following series.

3. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

4. $\dfrac{3}{1} + \dfrac{3}{2} + \dfrac{3}{6} + \dfrac{3}{24} + \dfrac{3}{120} + \cdots$

Examples: Write an expression for the following series using sigma notation.

5. $\dfrac{2}{1} + \dfrac{4}{3} + \dfrac{6}{5} + \dfrac{8}{7} + \cdots =$

6. $2 - 6 + 18 - 54 + 162 + \cdots =$

Example 7: What happens if we add more and more terms of a series like

$\sum_{n=0}^{\infty} (2n+1) = 1 + 3 + 5 + 7 + \cdots$ Show a sequence of partial sums.

This sequence of partial sums is approaching infinity. When this happens, the series is called **divergent**.

Example 8: What happens if we add more and more terms of $\sum_{n=1}^{\infty} \dfrac{3}{10^n} =$

This is an example of a **convergent** geometric series.

Geometric Series:
If consecutive terms in a series have a common ratio r, the series is called a **geometric series**.

$\sum_{n=0}^{\infty} a r^n = a + a r + a r^2 + \cdots + a r^n + \cdots$ is the general form of a geometric series.

If the geometric series converges it is possible to find the sum even though it has infinitely many terms.

Let $S = a + a r + a r^2 + a r^3 + \cdots$

then $rS = \quad a r +$

subtracting $S - rS =$

factoring $S(1-r) =$

$S =$

first term!

$$\sum_{n=0}^{\infty} a\,r^n = \frac{a}{1-r}$$ if the geometric series converges.

The Geometric Series Test: for a geometric series $\sum_{n=0}^{\infty} a\,r^n$

If $|r| \geq 1$ the geometric series **diverges**. If $|r| < 1$ the series **converges**.

Examples: Determine if these series converge or diverge and, if possible, find the sum of the series.

9. $\sum_{n=0}^{\infty} \frac{3}{2^n}$

10. $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

11. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^n$

Example 12: Find the fraction form of the repeating decimal $.\overline{08}$ using a geometric series.

Example 13: The series $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right) = (1+1)+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)+\cdots$ does **not** converge.

Show a sequence of partial sums.

A series **cannot** converge unless the terms approach a limit of **zero**.

***n*^th^ Term Test for <u>Divergence</u>:**

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. This test is inconclusive if $\lim_{n\to\infty} a_n = 0$.

Example 14: Show that $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ diverges.

Assignment 9.1

Simplify without using a calculator.

1. $\dfrac{7!}{10!}$ 2. $\dfrac{(2n+1)!}{(2n-1)!}$

Write an expression for the *nth* term of each series. Use $n = 1,2,3,\cdots$

3. $\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\cdots$ 4. $-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\cdots$ 5. $\frac{2}{1}+\frac{4}{3}+\frac{8}{7}+\frac{16}{15}+\cdots$ 6. $\frac{3}{1}+\frac{3}{2}+\frac{3}{6}+\frac{3}{24}+\cdots$

Use sigma notation to write an equivalent expression for each series. Use $n = 0,1,2,3,\cdots$

7. $\frac{1}{2}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\cdots$ 8. $-1+1+3+5+\cdots$ 9. $\frac{1}{1}+\frac{4}{3}+\frac{9}{9}+\frac{16}{27}+\cdots$

Determine whether each of the following infinite series converges or diverges. Show justification and name the test being used. In addition, find the sum of the series, if possible.

10. $\displaystyle\sum_{n=0}^{\infty}5\left(\frac{2}{3}\right)^n$ 11. $\displaystyle\sum_{n=1}^{\infty}5\left(\frac{2}{3}\right)^n$ 12. $\displaystyle\sum_{n=0}^{\infty}5\left(\frac{3}{2}\right)^n$ 13. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+2}{n^2}$

14. $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{\ln n}$ 15. $5+\frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\cdots$ 16. $3+\frac{9}{2}+\frac{27}{4}+\frac{81}{8}+\cdots$ 17. $1+0.2+0.04+0.008+\cdots$

18. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sin n}$ 19. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+2}{3^{n+2}}$ 20. $\displaystyle\sum_{n=0}^{\infty}\frac{e^n}{\pi^{n+1}}$ 21. $\frac{1}{2}+\frac{2}{4}+\frac{6}{8}+\frac{24}{16}+\frac{120}{32}+\cdots$

22. $\displaystyle\sum_{n=1}^{\infty}\left(\sin e^{10}\right)^n$ 23. $\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^n}{6}$ 24. $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{5}{6}\right)^n$ 25. $18-12+8-\frac{16}{3}+\frac{32}{9}-\cdots$

26. $\displaystyle\sum_{n=1}^{\infty}\frac{2n+3}{3n+2}$ 27. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{e^n}$ 28. $\displaystyle\sum_{n=0}^{\infty}\frac{4}{3^n}$ 29. $\displaystyle\sum_{n=1}^{\infty}4^{-n}$

30. Given the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+2}{n^3}$:

   a. Find $\lim\limits_{n\to\infty} a_n$.

   b. Explain why the *nth* Term Test **cannot** be used to conclude the series converges.

31. Find the x- and y-intercepts, relative extrema, and points of inflection for $y=\sin x+\cos x$ on $[0,2\pi)$. Then sketch the graph of y without using a calculator.

Differentiate implicitly to find $\frac{dy}{dx}$.

32. $\cos(y-x)=x^3+2$ 33. $2xy=\tan y^2$

Find the limits without using a calculator.

34. $\displaystyle\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos x}{x-\frac{\pi}{2}}\right)$ 35. $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{\sin(-5x)}$ 36. $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{x}$ 37. $\displaystyle\lim_{t\to 0}(2t\sec t)$

38. Without using a calculator find $\int_0^{\infty} \frac{e^x}{1+e^x}\,dx$.

For Problems 39-42, a region is in the 1st quadrant bounded by $y = 3\cos(2x)$, $y = 3x$, and $x = 0$.

39. Use a calculator to find the area of the region. Show an integral set up and an answer

40. Set up (but do not integrate) an integral for finding the volume of the solid formed by revolving the region about the x-axis.

41. Set up (but do not integrate) an integral for finding the volume of the solid formed by revolving the region about $y = 4$.

42. Set up (but do not integrate) an integral for finding the volume of the solid formed by using rectangular cross sections whose bases are in the region and are perpendicular to the x-axis. The heights of the rectangles are always half of their bases.

Selected Answers:

1. $\frac{1}{720}$ 4. $\frac{(-1)^n}{n^2}$ 6. $\frac{3}{n!}$ 7. $\sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}$ 10. converges by GST, Sum = 15

11. conv. by GST, Sum = 10 12. div. by GST or nTT 14. div. by nTT

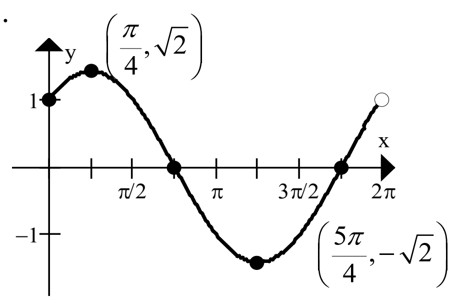
15. conv. by GST, Sum = 10 16. div. by GST or nTT 17. conv. by GST, Sum = 1.25

18. div. by nTT 20. conv. by GST, Sum = $\frac{1}{\pi - e}$ 22. conv. by GST, Sum = -.407 or -.408

23. div. by GST or nTT 24. conv. by GST, Sum = $-\frac{5}{11}$ 26. div. by nTT

27. div. by nTT 29. conv. by GST, Sum = $\frac{1}{3}$

31.



32. $y' = \frac{3x^2}{-\sin(y-x)} + 1 = \frac{3x^2 - \sin(y-x)}{-\sin(y-x)}$

33. $y' = \frac{-2y}{2x - 2y\sec^2 y^2}$ 34. -1 36. 3

38. diverges 39. .888 41. $\pi \int_0^{.5149} \left((4-3x)^2 - (4 - 3\cos(2x))^2 \right) dx$

42. $\int_0^{.5149} \frac{1}{2} \left(3\cos(2x) - 3x \right)^2 dx$